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On the spectral characterization of T-shape trees ☆

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Abstract

A graph G is said to be *determined by its spectrum* if any graph having the same spectrum as G is isomorphic to G. A *T-shape* is a tree with exactly one of its vertices having maximal degree 3. In this paper, we show that all T-shape trees are determined by their spectra, except for a few well-defined cases.

AMS classification: 05C50

Keywords: Spectra of graphs; Cospectral graphs; Spectral radius; T-shape trees; Determined by the spectrum

1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V = \{1, 2, \ldots, n\}$ and edge set E. *The adjacency matrix* $A = (a_{ij})$ of the graph G is an $n \times n$ matrix, where $a_{ij} = 1$ if i and j are adjacent; $a_{ij} = 0$ otherwise. The characteristic polynomial of the adjacency matrix A is called *the characteristic polynomial of the graph* G and is denoted by $\phi(G, \lambda)$ or simply $\phi(G)$. *The spectrum* of G, denoted by $\sigma(G)$, consists of the roots (together with their multiplicities) $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$ of the equation $\phi(G, \lambda) = 0$. The largest root $\lambda_1(G)$ is referred to as *the spectral radius* of G.

Two graphs are *cospectral* if they share the same spectrum. A graph G is said to be *determined by its spectrum* (DS for short) if for any graph H, $\phi(H, \lambda) = \phi(G, \lambda)$ implies that H is isomorphic to G.

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Determining what kinds of graphs are DS is an old problem, yet far from resolved, in the theory of graph spectra. Numerous examples of cospectral but non-isomorphic graphs are reported in literature (see Chapter 6 in [1] for instance). However, there are few results known about DS graphs. For the background and some recent surveys of the known results about this problem and related topics, we refer the reader to [2,7] and the references therein.

Because the kind of problems above are generally very hard to deal with, some more modest ones are suggested by van Dam and Haemers [2], say, "which trees are DS?". This paper will give a complete answer to this modified problem for a class of specific trees.

A tree is *starlike* if exactly one of its vertices has degree larger than 2, and *T-shape* if it is starlike with maximal degree 3. We will denote by $T(l_1, l_2, l_3)$ (assume $l_1 \leqslant l_2 \leqslant l_3$ without loss of generality in the sequel) the unique T-shape tree such that $T(l_1, l_2, l_3) - v = P_{l_1} \bigcup P_{l_2} \bigcup P_{l_3}$, where P_{l_i} is the path on l_i vertices ($i = 1, 2, 3$), and v the vertex of degree 3.

Lepović and Gutman [6] proved that no two starlike trees share the same spectrum. This is rather unexpected due to the famous result of Schwenk [8], which says that almost all trees have non-isomorphic cospectral mates. More recently, the first author [10] proved that $T(1, l_2, l_3)$ is determined by its spectrum. In this paper, we further show that all T-shape trees are DS, except for a few well-defined cases. More precisely, the main result of the paper is as follows:

Theorem 1.1. *Let $G = T(l_1, l_2, l_3)$ $(l_1 \leqslant l_2 \leqslant l_3)$. Then G is uniquely determined by its spectrum iff $(l_1, l_2, l_3) \neq (l, l, 2l - 2)$ for any positive integer $l \geqslant 2$.*

The proof of the theorem above is, however, somewhat technical which mainly involves several eigenvalue properties of graphs and a directly comparing of the characteristic polynomials of the graphs concerned (under suitable transforms of the variable).

2. Some lemmas

In this section, we will present some lemmas which are required in the proof of the main result. First we give some known results about the spectra of graphs.

Lemma 2.1 [1]. *Let G be a connected graph, and H a proper subgraph of G. Then $\lambda_1(H) < \lambda_1(G)$.*

Hoffman and Smith [4] define an internal path of a graph G as a walk $v_0, v_1, \ldots, v_k$ $(k \geqslant 1)$ such that the vertices $v_1, \ldots, v_k$ are distinct (v_0, v_k need not be distinct), $\deg(v_0) > 2$, $\deg(v_k) > 2$ and $\deg(v_i) = 2$ for $0 < i < k$. The following lemma will be repeatedly used in the paper.

Lemma 2.2 [4]. *Let G be a connected graph that is not isomorphic to W_n, where W_n is a graph obtained from the path P_{n-2} (indexed in natural order $1, 2, \ldots, n-2$) by adding two pendant edges at vertices 2 and $n-3$. Let G_{uv} be the graph obtained from G by subdividing the edge uv of G. If uv lies on an internal path of G, then $\lambda_1(G_{\mathrm{uv}}) < \lambda_1(G)$.*

Lemma 2.3 [1]. *Let G be the graph obtained from the disjoint union $H_1 \bigcup H_2$ by adding an edge v_1v_2 joining the vertex v_1 of H_1 and v_2 of H_2, then*

$$\phi(G) = \phi(H_1)\phi(H_2) - \phi(H_1 - v_1)\phi(H_2 - v_2),$$

where $H_i - v_i$ denotes the graph obtained from H_i by deleting the vertex v_i and the edges incident to it ($i = 1, 2$).

Lemma 2.4 [1]. *Let C_n, P_n denote the cycle and the path on n vertices respectively. Then*

$$\phi(C_n,\lambda)=\prod_{j=1}^{n}\left(\lambda-2\cos\frac{2\pi j}{n}\right)=2\cos(n\arccos\lambda/2)-2,$$
$$\phi(P_n,\lambda)=\prod_{j=1}^{n}\left(\lambda-2\cos\frac{\pi j}{n+1}\right)=\frac{\sin((n+1)\arccos\lambda/2)}{\sin(\arccos\lambda/2)}.$$

Let $\lambda=2\cos\theta$, set $t^{1/2}=\mathrm{e}^{\mathrm{i}\theta}$, it is useful to write the characteristic polynomial of C_n, P_n in the following form:

$$\phi(C_n,t^{1/2}+t^{-1/2})=t^{n/2}+t^{-n/2}-2, \tag{1}$$

$$\phi(P_n,t^{1/2}+t^{-1/2})=t^{-n/2}(t^{n+1}-1)/(t-1). \tag{2}$$

Lemma 2.5 [1]. *Let $\phi(G,\lambda)=\lambda^n+a_1\lambda^{n-1}+\cdots+a_n$ be the characteristic polynomial of graph G. The coefficient a_i is equal to $\sum_\gamma(-1)^{comp(\gamma)}2^{cyc(\gamma)}$, where the sum is taken over all subgraphs γ consisting of disjoint edges and cycles, and having i vertices; $comp(\gamma)$ is the number of components, and $cyc(\gamma)$ the number of cycles, of γ.*

Denote by $M_2(G)$ the set of all 2-matchings in a graph G (a *k-matching* consists of k independent edges in $E(G)$ which are pairwise non-adjacent), then it follows immediately that $a_4=|M_2(G)|$ provided that G contains no cycle of length 4. Thus, if G and H are two cospectral graphs without cycles of length 4, then $|M_2(G)|=|M_2(H)|$. This fact will be frequently used in the sequel.

Lemma 2.6 [6]. *No two non-isomorphic starlike trees have the same spectrum.*

Lemma 2.7 (cf. [5]). *Let λ_1 be the spectral radius of $T(l_1,l_2,l_3)$, then $\lambda_1<\frac{3}{\sqrt{2}}$.*

Proof. Let m be a positive integer such that $l_j<m(j=1,2,3)$. Denote $T_m=T(m,m,m)$, it follows from Lemma 2.3 that $\phi(T_m,\lambda)=(\phi(P_m,\lambda))^2(\lambda\phi(P_m,\lambda)-3\phi(P_{m-1},\lambda))$. By (2) we have

$$\phi(T_m,t^{1/2}+t^{-1/2})/(\phi(P_m,\lambda))^2=\frac{t^{-(m+1)/2}}{t-1}(t^{m+2}-2t^{m+1}+2t-1)=:\psi(t). \tag{3}$$

Let t_m be the largest root of $\psi(t)$, then $t_m<2$ since $\psi(t)>0$ for $t\geqslant 2$. Let $f(t)=t^{1/2}+t^{-1/2}$, then $f'(t)=t^{-3/2}(t-1)/2\geqslant 0$ for $t\geqslant 1$, so $f(t)$ strictly increases in $[1,\infty)$. Thus $\lambda_1(T_m)=t_m^{1/2}+t_m^{-1/2}<2^{1/2}+2^{-1/2}=\frac{3}{2}\sqrt{2}$. Moreover, by Lemma 2.1 we have $\lambda_1(T(l_1,l_2,l_3))<\lambda_1(T_m)$, thus the lemma holds. □

Lemma 2.8 (cf. [9]). *Let $G=T(l_1,l_2,l_3)$ be a T-shape tree. Then $2\in\sigma(G)$ iff $(l_1,l_2,l_3)=(2,2,2),(1,2,5)$ or $(1,3,3)$.*

Proof. Smith [9] gives all graphs with spectral radius 2: the circle C_n, the graph W_n (defined as in Lemma 2.2), the T-shape tree $T(2,2,2)$, $T(1,2,5)$ and $T(1,3,3)$. By eigenvalue interlacing property, we have $\lambda_2(G)\leqslant\lambda_1(G-v)$, where v is the vertex of degree 3 of the T-shape tree G. Moreover, it is obvious that $\lambda_1(G-v)<2$. It follows that $\lambda_2(G)<2$, and $2\in\sigma(G)$ implies that $\lambda_1(G)=2$. So the lemma follows immediately from Smith's result. □

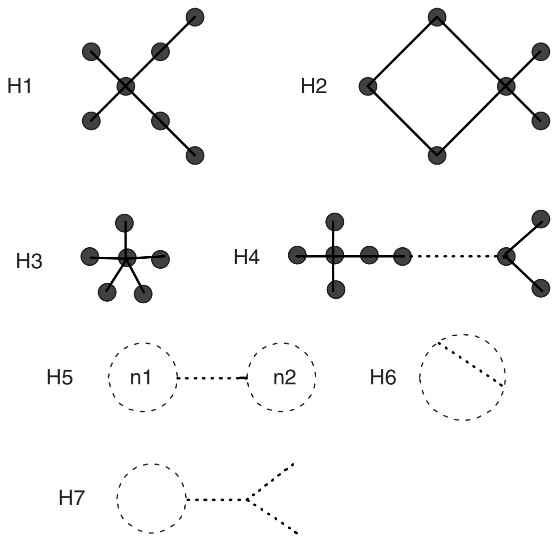


Fig. 1. Forbidden subgraphs of H.

Recall that a graph Γ is called a *forbidden subgraph* of a graph G, if no subgraph of G is isomorphic to Γ. The following lemma gives some forbidden subgraphs of a graph H that is cospectral with a T-shape tree.

Lemma 2.9. *Let $G = T(l_1, l_2, l_3)$ be a T-shape tree, and let H be cospectral with G. Then the following graphs H_i $(i = 1, \ldots, 7)$ are forbidden subgraphs of H, where H_5 is obtained from the disjoint union of two cycles by adding a path (which is allowed to contract to a point) connecting a vertex of one cycle and that of the other; H_6 is obtained from a cycle by connecting two of its vertices by a path; H_7 is obtained from the disjoint union of a cycle and a T-shape tree by identifying a vertex of the cycle with a vertex of degree 1 of the T-shape tree (Fig. 1).*

Proof. Direct computation shows $\lambda_1(H_1) = 2.1357\cdots > \frac{3}{\sqrt{2}} = 2.1213\cdots > \lambda_1(G)$; $\lambda_1(H_2) = 2.2882\cdots > \frac{3}{\sqrt{2}} = 2.1213\cdots > \lambda_1(G)$; $\lambda_1(H_3) = \sqrt{5} > \frac{3}{\sqrt{2}} > \lambda_1(G)$. It follows from Lemma 2.1 that H_1, H_2 and H_3 are forbidden subgraphs of H.

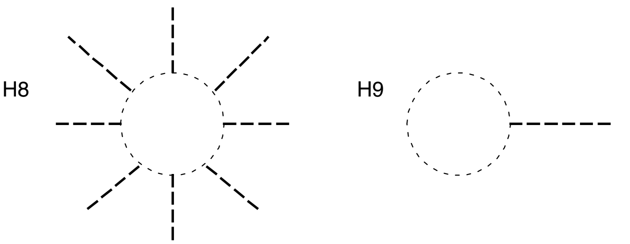
Moreover, it is easy to see that H_i $(i = 5, 6, 7)$ are not isomorphic to W_k for any k. Note that one can always subdivide certain edges which lie on an internal path of H_i successively in an appropriate way, to obtain graph $\widetilde{H_i}$, such that G can be embedded in $\widetilde{H_i}$ as a proper subgraph $(i = 5, 6, 7)$. Let us take graph H_5 for example. One can subdivide the cycle C_{n_1} and the path between C_{n_1} and C_{n_2} (both are internal paths, and if the path contracts to a point we subdivide C_{n_1} and C_{n_2} instead) successively such that the length of the former $> l_1 + l_2$, and the length of the latter $> l_3$. Then it is clear that G is isomorphic to a proper subgraph of $\widetilde{H_5}$. Using this way, we have $\lambda_1(H_i) > \lambda(\widetilde{H_i}) > \lambda_1(G)$, from which we get H_5, H_6 and H_7 are forbidden subgraphs of H. Next, it needs to show that H_4 is a forbidden subgraph of H.

Suppose the order of H_4 is n, using Lemma 2.3 recursively, we obtain $\phi(H_4, \lambda) = \lambda^3(\phi(P_{n-3}) - 3\phi(P_{n-5}) + 2\phi(P_{n-7}))$. So $\lambda_1(H_4)$ is the largest root of $\phi(H_4, \lambda)/\lambda^3 = 0$, which is equivalent to the following equation in t according to (2):

$$\varphi(t) := t^{n-2} - 3t^{n-3} + 2t^{n-4} - 2t^2 + 3t - 1 = 0. \tag{4}$$

Let t_n be the largest root of $\varphi(t)$, since $\varphi(2) = -3 < 0$, it follows that $t_n > 2$. Thus, using the fact that $t^{1/2} + t^{-1/2}$ strictly increases in $[1, \infty)$ we get $\lambda_1(H_4) = t_n^{1/2} + t_n^{-1/2} > 2^{1/2} + 2^{-1/2} = \frac{3}{\sqrt{2}} > \lambda_1(H)$. This completes the proof. □

In what follows, $\Gamma(m; m_1, \cdots, m_k; v_1, \ldots, v_k)(m_i \geqslant 1, i = 1, \ldots, k)$, depicted as H_8, is the graph obtained from C_m by adding $k(\geqslant 0)$ pendant paths of length $m_1, \ldots, m_k$ at vertices $v_1, \cdots, v_k$ respectively, which will be briefly denoted as $\Gamma(m, k)$ if no confusion arises. In particular, if $k = 1$, $D(m_1, m_2)(m_2 \geqslant 3, m_1 \geqslant 0)$, depicted as H_9, will denote the graph obtained from $C_{m_2} \bigcup P_{m_1}$ by adding an edge to connect a vertex of C_{m_2} and a vertex of degree 1 in P_{m_1}.



The following lemma gives the number of all 2-matchings in a graph G with maximal degree 3, in terms of the number of edges and the number of vertices of degree 3, of G.

Lemma 2.10. *Let T be a tree with maximal degree $\Delta = 3$ on $n(\geqslant 2)$ vertices. Then $|M_2(T)| = ((n-1)^2 - 3(n-1) - 2(k-1))/2$, where k is the number of vertices with degree 3 of T (we allow that $k = 0$ for convenience). Moreover, let $\Gamma(m, k)(m \geqslant 5)$ be the graph defined as above on n vertices, then $|M_2(\Gamma(m, k))| = (n^2 - 3n - 2k)/2$.*

Proof. The second assertion of the lemma can be easily deduced from the first one, as we will see later. So we concentrate on the proof of the first one. The cases $k = 0$ and $T = K_{1,3}$ are trivial, so we assume $k \geqslant 1$ and $T \neq K_{1,3}$ in the sequel.

Let $e \in E(T)$ be an edge in T, then e is adjacent to either 1, 2, 3 or 4 edges in $E(T)$. Let E_i be the set of edges that are adjacent to exactly i edges in T $(i = 1, 2, 3, 4)$. Let $|E_i| = m_i$, then we have $m_1 + m_2 + m_3 + m_4 = n - 1$. For a fixed edge e in E_i, the number of edges which are non-adjacent to e is $n - 1 - (i + 1)$. Thus, the number of all 2-matchings in T is $|M_2(T)| = \frac{1}{2}\sum_{i=1}^{4} m_i(n - 1 - (i + 1))$. Substitute $m_2 = n - 1 - m_1 - m_3 - m_4$ in the equality above, we obtain $|M_2(T)| = \frac{1}{2}((n-1)^2 - 3(n-1) - (2m_4 + m_3 - m_1))$. So it needs only to show

$$2m_4 + m_3 - m_1 = 2(k - 1). \tag{5}$$

Without loss of generality, we will assume that $m_1 = 0$ and $m_4 = 0$ in the remaining proof. This is because eliminating edges in E_1 and E_4 gives no any change in (5). In fact, delete one edge in E_1, either m_1 is unchanged, or m_1 and m_3 reduce 1 simultaneously. Similarly, for $e \in E_4$, subdividing e generates a new tree T'. This operation also gives no change in (5), since m_4 reduces 1, while m_3 increases by 2.

With the assumption $E_1, E_4 = \emptyset$ in $E(T)$, each vertex of degree 3 in T is adjacent to either 0, 1 or 2 leaves (a vertex of degree 1), and any two vertices of degree 3 are not adjacent. Let k_i be the number of vertices of degree 3 having exactly $3 - i$ $(i = 1, 2, 3)$ leaves, then the vertex of degree 3 having $3 - i$ leaves is incident to i edges in E_3, so $m_3 = k_1 + 2k_2 + 3k_3$. ($E_4 = \emptyset$

implies that no repeats can occur when we are counting the edges in E_3.) So it suffices to prove $m_3 = k_1 + 2k_2 + 3k_3 = 2(k-1) = 2(k_1 + k_2 + k_3 - 1)$, that is

$$k_1 - k_3 = 2. \tag{6}$$

Furthermore, note that a deletion of the leaf adjacent to a vertex of degree 3 having exactly one leaf does not change (6). So we can also assume that T has no vertex of degree 3 with exactly one leaf, i.e., $k_2 = 0$. Now, let v be a vertex of T with 2 leaves, and w the vertex of T of degree 3 which is nearest to v. There are two possible cases: (a) w has exactly two leaves; (b) w has no leaves. For case (a), it is clear that $k_1 = 2$, $k_3 = 0$, so (6) holds. Under case (b), deleting the path between v and w, together with the two leaves of v and the isolated vertices, we will obtain a new tree T'' and k_1, k_3 will decrease by 1 simultaneously for T''. So it suffices to prove that (6) holds for T''. Applying the same argument to T'', ..., eventually we will return to case (a) since the order of T is finite, and each operation in case (b) will reduce the order of trees strictly. Hence (6) holds and the first assertion is proved.

Regarding to the second assertion, let e be an edge of $\Gamma(m,k)$ which is on the cycle C_m and adjacent to an edge of a pendant path. Deleting e we obtain a tree $\widetilde{T}$ having either $k-1$ or $k-2$ vertices of degree 3, and the number of 2-matchings in $\Gamma(m,k)$ equals the number of 2-matchings in $\widetilde{T}$ plus the number of edges in $\widetilde{T}$ which is not adjacent to e. In the former case we obtain:

$$|M_2(\Gamma(m,k))| = |M_2(\widetilde{T})| + (n-4) = (n^2 - 3n - 2k)/2.$$

For the latter, the same result can also be obtained in a similar way. The proof is complete. □

3. Proof of the main result

Now we are ready to prove the the following lemma, which is the key to the proof of the main result of the paper.

Lemma 3.1. *Let $G = T(l_1, l_2, l_3)(l_1 \leqslant l_2 \leqslant l_3)$, let $H = D(m_1, m_2) \bigcup P_{m_3}$. Then H is cospectral with G iff $(l_1, l_2, l_3) = (l, l, 2l-2)$ and $(m_1, m_2, m_3) = (l-2, 2l+2, l-1)$ for some positive integer $l \geqslant 2$.*

Proof. The characteristic polynomials of G and H can be computed by Lemma 2.3 as follows:

$$\phi(G,\lambda) = \phi(P_{l_1})\phi(P_{l_2})\phi(P_{l_3})\left(\lambda - \frac{\phi(P_{l_1-1})}{\phi(P_{l_1})} - \frac{\phi(P_{l_2-1})}{\phi(P_{l_2})} - \frac{\phi(P_{l_3-1})}{\phi(P_{l_3})}\right), \tag{7}$$

$$\phi(H,\lambda) = (\phi(C_{m_2},\lambda)\phi(P_{m_1},\lambda) - \phi(P_{m_2-1},\lambda)\phi(P_{m_1-1},\lambda))\phi(P_{m_3},\lambda). \tag{8}$$

By (1) and (2) we can write (using Mathematica 5.0):

$$\begin{aligned}
&\phi(G, t^{1/2} + t^{-1/2})t^{(l_1+l_2+l_3+1)/2}(t-1)^3 \\
&\quad = t^{l_1+l_2+l_3+4} - 2t^{l_1+l_2+l_3+3} + t^{l_2+l_3+2} + t^{l_1+l_3+2} \\
&\qquad + t^{l_1+l_2+2} - t^{l_3+2} - t^{l_2+2} - t^{l_1+2} + 2t - 1,
\end{aligned} \tag{9}$$

$$\begin{aligned}
&\phi(H, t^{1/2} + t^{-1/2})t^{(m_1+m_2+m_3)/2}(t-1)^3 \\
&\quad = t^{m_1+m_2+m_3+3} - 2t^{m_1+m_2+m_3+2} + t^{m_2+m_3+1}
\end{aligned}$$

$$\begin{aligned}&-2t^{m_1+m_2/2+m_3+3}+2t^{m_1+m_2/2+m_3+2}+2t^{m_2/2+m_3+2}-2t^{m_2/2+m_3+1}+t^{m_1+m_3+3}\\&-2t^{m_3+2}+t^{m_3+1}-t^{m_1+m_2+2}+2t^{m_1+m_2+1}-t^{m_2}+2t^{m_2/2}+2t^{m_1+m_2/2+2}\\&-2t^{m_1+m_2/2+1}-2t^{m_2/2+1}-t^{m_1+2}+2t-1.\end{aligned}\tag{10}$$

If $(l_1, l_2, l_3) = (l, l, 2l-2)$ and $(m_1, m_2, m_3) = (l-2, 2l+2, l-1)$. A straightforward computation shows that

$$\begin{aligned}\phi(G, t^{1/2}+t^{-1/2}) &= t^{-(m_1+m_2+m_3)/2}(t-1)^{-3}\\&\quad\times(t^{4l+2}-2t^{4l+1}+2t^{3l}+t^{2l+2}-t^{2l}-2t^{l+2}+2t-1),\\\phi(H, t^{1/2}+t^{-1/2}) &= t^{-(l_1+l_2+l_3+1)/2}(t-1)^{-3}\\&\quad\times(t^{4l+2}-2t^{4l+1}+2t^{3l}+t^{2l+2}-t^{2l}-2t^{l+2}+2t-1).\end{aligned}$$

Clearly

$$l_1+l_2+l_3+1 = m_1+m_2+m_3.\tag{11}$$

So G and H are cospectral.

Now suppose that G and H are cospectral. It follows that the right sides of (9) and (10) (denote by ϕ_1 and ϕ_2 respectively) must be identical. Next, we will prove the necessity of the lemma by comparing the coefficients of the corresponding terms of ϕ_1 and ϕ_2. Note that the first two and the last two terms of ϕ_1 and that of ϕ_2 are identical, so they can be subtracted from ϕ_1 and ϕ_2 simultaneously (the other terms of ϕ_1 and ϕ_2 are left unchanged). Using the same notation ϕ_1, ϕ_2 after subtractions, then it is enough to compare the terms of the new ϕ_1 and ϕ_2.

First it is easy to see $m_1 \leqslant l_1 - 1$. Otherwise by subdividing the edges of C_{m_2} successively, we will obtain a graph $D(m_1, m_2')$ such that G can be embedded in $D(m_1, m_2')$ as a proper induced subgraph. It follows from Lemma 2.2 that $\lambda_1(H) > \lambda_1(D(m_1, m_2')) > \lambda_1(G)$; a contradiction. Also, it is clear that m_2 is even, and we can assume $m_2 \geqslant 6$ because

$$\lambda_1(D(m_1, 4)) \geqslant \lambda_1(D(1, 4)) = 2.1357\cdots > \frac{3}{\sqrt{2}} > \lambda_1(G).\tag{12}$$

Observe that the exponents of the terms of ϕ_1 are arranged in non-increasing order, except possibly that the exponent of the 4th term may be larger than that of the 3rd term. Now consider the first term, $t^{m_2+m_3+1}$, of ϕ_2. By (11) and the relation $m_1 \leqslant l_1 - 1$, it is easy to show that $m_2 + m_3 + 1 > l_2 + l_3 + 2$. Thus the term $t^{m_2+m_3+1}$ must vanish in ϕ_2, that is, there must exist the other terms of the same exponents in ϕ_2, the sum of the coefficients of which plus 1 equals zero. It is clear that all the possible candidates are the 2nd, 3rd, 6th, 9th, 10th, 13th and 14th terms of ϕ_2, since the exponents of the rest terms are either larger than or less than $m_2 + m_3 + 1$. We consider the following possible cases:

Case 1. If either the 2nd or the 3rd term of ϕ_2 is chosen. Note that the exponent of the 2nd or the 3rd term is larger than that of the terms behind it except possibly for the 9th and 10th terms, so all the possible combinations of terms are: {1st, 2nd}, {1st, 3rd}, {1st, 2nd, 9th}, {1st, 2nd, 10th}, {1st, 3rd, 9th} and {1st, 3rd, 10th}. But all these choices are impossible since the sum of the coefficients does not vanish for each choice.

Case 2. If the 6th term is chosen. In order for the coefficient of $t^{m_2+m_3+1}$ in ϕ_2 to be zero, one can only choose the 14th term. This gives $1 + m_2 + m_3 = 3 + m_1 + m_3 = 1 + m_1 + m_2/2$, so we obtain $m_1 + m_2 + m_3 = 5/2m_1 + 1 = l_1 + l_2 + l_3 + 1 \geqslant 3l_1 + 1$. It follows that $m_1 \geqslant 6/5l_1 > l_1$, which is a contradiction.

Case 3. If either the 13th or the 14th term is chosen. The same argument as case 1 can be used to get a contradiction.

Case 4. So the only choice left is the 9th term (the 10th term is obviously impossible), which gives $1 + m_2 + m_3 = 2 + m_1 + m_2$, that is, $m_3 = m_1 + 1$. By such a choice, the term $t^{m_2+m_3+1}$ vanishes in ϕ_2.

Now, cancel the identical terms in ϕ_1 and ϕ_2 simultaneously (the new polynomials of t are still denoted by ϕ_1 and ϕ_2), then substitute $m_3 = m_1 + 1$, we obtain

$$\begin{aligned}\phi_1 &= t^{l_2+l_3+2} + t^{l_1+l_3+2} + t^{l_1+l_2+2} - t^{l_3+2} - t^{l_2+2} - t^{l_1+2},\\ \phi_2 &= -2t^{2m_1+m_2/2+4} + 2t^{2m_1+m_2/2+3} + 2t^{m_1+m_2/2+3} + t^{2m_1+4} - 2t^{m_1+3} + 2t^{m_1+m_2+1}\\ &\quad - t^{m_2} + 2t^{m2/2} - 2t^{m_1+m_2/2+1} - 2t^{m_2/2+1}.\end{aligned}$$

Suppose first $m_1 < l_1 - 1$, then $m_1 + 3 < l_1 + 2$, it follows that the term t^{m_1+3} must vanish in ϕ_2. All the possible terms having the same exponents as t^{m_1+3} are the 7th, 8th, 9th and 10th terms of ϕ_2. Among them, only the 8th term, $2t^{m_2/2}$, the sum of which and $-2t^{m_1+3}$ might be zero. This gives $m_1 + 3 = m_2/2$, i.e., $m_2 = 2m_1 + 6$. Substitute it into ϕ_2, we get

$$\phi_2 = 2t^{3m_1+6} + t^{2m_1+6} - t^{2m_1+4} - 2t^{m_1+4}.$$

Comparing the lowest term of ϕ_1 and ϕ_2, it can be deduced that $l_1 + 2 = l_2 + 2 = m_1 + 4$, i.e., $l_1 = l_2, m_1 = l_1 - 2$. Comparing the first term of ϕ_1 and ϕ_2 gives $l_1 + l_3 + 2 = 3m_1 + 6$, that is, $l_3 = 2m_1 + 2 = 2l_1 - 2$. Let $l = l_1$, we obtain $(l_1, l_2, l_3) = (l, l, 2l - 2)$, $(m_1, m_2, m_3) = (l - 2, 2l + 2, l - 1)$, and

$$\phi_1 = \phi_2 = 2t^{3l} + t^{2l+2} - t^{2l} - 2t^{l+2}.$$

To complete the lemma, it remains to consider the case $m_1 = l_1 - 1$. Now since $m_1 + 3 = l_1 + 2$, we claim $l_1 = l_2$, this is because the coefficient of t^{m_1+3} in ϕ_2 either equals 0, -2, -3 or $\leqslant -4$; while the coefficient of t^{l_1+2} in ϕ_1 equals either -1 , -2 or -3 (corresponding to the cases $l_1 < l_2, l_1 = l_2 < l_3, l_1 = l_2 = l_3$ respectively). Again, cancel the identical terms in ϕ_1 and ϕ_2 simultaneously (using the same notations ϕ_1, ϕ_2), we obtain

$$\begin{aligned}\phi_1 &= 2t^{l_1+l_3+2} - t^{l_3+2},\\ \phi_2 &= -2t^{2m_1+m_2/2+4} + 2t^{2m_1+m_2/2+3} + 2t^{m_1+m_2/2+3} + 2t^{m_1+m_2+1}\\ &\quad - t^{m_2} + 2t^{m2/2} - 2t^{m_1+m_2/2+1} - 2t^{m_2/2+1}.\end{aligned}$$

Thus we must have $m_2 = l_3 + 2$, this can be easily seen by comparing $\phi_1(mod2)$ and $\phi_2(mod2)$. Therefore the exponents of all the terms except $-t^{m_2}$ in ϕ_2 must be equal, but it is obviously impossible. This completes the proof. □

Proof of the main result. The necessary condition follows immediately from Lemma 3.1. Now suppose that $(l_1, l_2, l_3) \neq (l, l, 2l - 2)$ for any integer $l \geqslant 2$ and H is a graph being cospectral with G. We proceed to prove that H must be isomorphic to G. Let Δ be the maximal degree of H, and $n = l_1 + l_2 + l_3 + 1$ the order of G. Since H_3 is a forbidden subgraph of H, we obtain $\Delta \leqslant 4$. We distinguish the following cases:

Case 1. If $\Delta = 4$. First we show that H contains no cycle. If not, suppose that H has a cycle. Let $C^1, C^2, \ldots, C^t$ be the connected components of H, assume that C^1 has a vertex of degree 4. The fact that H_4 is a forbidden subgraph of H implies that C^1 has exactly one vertex of degree

4, and the degree of the other vertices can only be 1 or 2. If C^1 contains a cycle (the length of which can be assumed to be even, and hence $\neq 3$, since H must be bipartite), then it must contain a subgraph like H_1 or H_2. This contradicts to Lemma 2.9, and hence C^1 has no cycle. It follows that there is another component, say C^2, which contains a cycle (of length m, say). From $\lambda_1(C^1) \geqslant \lambda_1(K_{1,4}) = 2, \lambda_1(C^2) \geqslant \lambda_1(C_m) = 2$ and $\sigma(H) = \bigcup_{i=1}^{t} \sigma(C^i)$, we get that there exist at least two eigenvalues of H no less than 2. This contradicts to the fact that H has exactly one eigenvalue no less than 2, since $\lambda_2(H) = \lambda_2(G) < 2$ (see the proof of Lemma 2.8). So H has no cycle. Moreover, H and G having the same number of edges forces H to be a tree. Again, using the fact that H_1 and H_4 are forbidden subgraphs of H , we get that H is such a tree that it can be obtained from P_{n-2} (indexed in a natural order as $1, 2, \ldots, n-2$) by adding two pendant edges at the vertex 2. It follows from Lemma 10 that $|M_2(G)| = ((n-1)^2 - 3(n-1))/2$, while a direct computation shows $|M_2(H)| = ((n-1)^2 - 3(n-1) - 4)/2$, which contradicts to Lemma 2.5.

Case 2. If $\Delta = 3$. We further consider two cases:

Subcase 1. H contains no cycle. Then $\phi(H) = \phi(G)$ implies that H and G have the same number of edges. It follows that H must be a tree. By Lemma 2.10, H and G have the same number of vertices of degree 3, i.e., H is a T-shape tree. Thus, H is isomorphic to G according to Lemma 2.6.

Subcase 2. H contains some cycles. By Lemma 2.9, H_5 and H_6 are forbidden subgraphs of H. It follows that each connected component of H contains at most one cycle. Using the same argument as in case 1, it can be shown that there exists exactly a connected component of H which contains a cycle. Since H and G have the same number of edges, we get that H is the disjoint union of a unicyclic graph U and a tree T_1. Moreover, H_7 is a forbidden subgraph of H, it follows that U must be a graph like $\Gamma(m, k)$.

Now let the number of vertices of degree 3 in T_1 be s, let n_1 and n_2 denote the number of edges of $\Gamma(m, k)$ and T_1 respectively. First suppose that $m = 4$. Note that $D(1, 4)$ is a forbidden subgraph of H according to equation (12), it follows that $\Gamma(m, k) \cong C_4$, and hence $n_2 \neq 0$. Now compare the coefficients of λ^{n-4} in $\phi(G, \lambda)$ and $\phi(H, \lambda)$ respectively. According to Lemma 2.5 and Lemma 2.10, we get $a_4(H) = |M_2(H)| - 2 = (n_2^2 + 5n_2 + 4 - 2(s-1))/2 - 2$ and $a_4(G) = |M_2(G)| = (n_2^2 + 5n_2 + 4)/2$. Thus we have $a_4(G) > a_4(H)$ since $s \geqslant 0$; a contradiction. Therefore in what follows, we can assume $m > 4$, i.e., $\Gamma(m, k)$ contains no cycle of length 4.

If $n_2 = 0$, it is clear that $|M_2(H)| = |M_2(\Gamma(m, k))|$. By Lemma 2.10 we get $|M_2(\Gamma(m, k))| = (n_1^2 - 3n_1 - 2k)/2$ and $|M_2(G)| = (n_1^2 - 3n_1)/2$. It follows from Lemma 2.5 that $|M_2(H)| = |M_2(G)|$. Thus we have $k = 0$. However, this contradicts to the assumption $\Delta = 3$.

So we can assume that $n_2 \geqslant 1$. By Lemma 2.10 we get

$$\begin{aligned} |M_2(H)| &= |M_2(\Gamma(m, k))| + |M_2(T_1)| + n_1 n_2 \\ &= \frac{1}{2}((n_1 + n_2)^2 - 3(n_1 + n_2) - 2(k + s - 1)). \end{aligned}$$

Moreover, $|M_2(G)| = \frac{1}{2}((n_1 + n_2)^2 - 3(n_1 + n_2))$. Comparing $|M_2(H)|$ and $|M_2(G)|$ gives $k + s = 1$, i.e.,

either $k = 1, s = 0$ or $k = 0, s = 1$.

If $k = 1, s = 0$, then $H = D(m_1, m_2) \bigcup P_{m_3}$ for some m_i $(i = 1, 2, 3)$. It follows from Lemma 3.1 that H and G are not cospectrtal since $(l_1, l_2, l_3) \neq (l, l, 2l - 2)$ for any integer$l \geqslant 2$, which is a contradiction.

If $k = 0, s = 1$, then $\Gamma(m, k) \cong C_m$ (m is even, and hence $\geqslant 6$) and T_1 is a T-shape tree. It is clear that $2 \in \sigma(G)$. By Lemma 2.8, $G = G_1 := T(1, 2, 5)$ or $G = G_2 := T(1, 3, 3)$ ($T(2, 2, 2)$ is excluded by the assumption of the theorem). Note that the order of H is no less than 10, but the orders of G_1 and G_2 are 9 and 8 respectively. It is impossible.

Case 3. If $\Delta = 2$, then H either is a path P_n or the disjoint union of a cycle $D(0, m)(\cong C_m)$ and a path $P_{m'}$. In the former case, it follows from Lemma 2.10 that $|M_2(G)| = ((n-1)^2 - 3(n-1))/2$ and $|M_2(H)| = |M_2(G)| + 1$, it is a contradiction. In the latter case, by Lemma 3.1 we get $m = 6, m' = 1$ and $(l_1, l_2, l_3) = (2, 2, 2)$. It contradicts to the previous assumption.

Case 4. $\Delta = 1$ is obviously impossible.

Combing cases 1–4, H is isomorphic to G. The proof is complete. $\square$

4. An open problem

We end the paper by proposing the following problem which generalizes [3]:

Conjecture 1. *Let $G = T(l_1, l_2, l_3)(l_1 \leqslant l_2 \leqslant l_3)$ be a T-shape tree. Then $\bar{G}$ (the complement of G) is DS iff $(l_1, l_2, l_3) \neq (l, l, 2l - 2)$ for any positive integer $l \geqslant 2$.*

In fact, it is not difficult to show that $\bar{G}$ and $\bar{H}$ are cospectral if $(l_1, l_2, l_3) = (l, l, 2l - 2)$, where $H = D(l - 2, 2l + 2) \bigcup P_{l-1}$. However, it seems that there is no obvious way to extend the method in this paper to solve the problem above.

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